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PRESENTE

Por medio de la presente nos permitimos informarle que su trabajo: Technological Catching up across Countries in Bio-Pharmaceutical F ield. A Volterra Integro-Differential Equations Application, 1980-2010, fue recibido en el mes de agosto de 2015 y aceptado en el mismo año después de un doble arbitraje ciego para formar parte del libro: *Modelado de Fenómenos Económicos y F inancieros: Una Visión Contemporánea, Volumen 1, págs 371-401,* editado por la Universidad de las Américas Puebla, la Universidad Nacional Autónoma de México, el Instituto Politécnico Nacional y Editorial UDLAP, coordinado por la Dra. Claudia Estrella Castillo Ramírez, el Dr. Francisco López Herrera y el Dr. Francisco Venegas Martínez.

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ATENTAMENTE

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Dra. Claudia Estrella Castillo Ramírez

Índice

TECHNOLOGICAL AND INNOVATION GAPS. A VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS (VIDE) APPLICATION TO THE ANALYSIS OF PATENT ACROSS COUNTRIES

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Abstract

The aim of this research is to model the heredity principle of the Volterra integro-differential equations (*VIDE*) in the case of sectoral technological catching-up favoring its economic growth. We work on the assumption of a "non heredity principle" when considering that the future of a system for a determined moment only depends on its current state. This is a restrictive hypothesis, as sometimes, the future of a system seems to be dependent on primary states, as it is the case of the heredity. The functional form of the equation is one in which appears a convolution integral. The latter represents the hereditary part. The problem we are faced with refers to the process by which countries can benefit from the existence of a knowledge production stock that is available in this field of expertise in the rest of the world for a determined period of time.

We pose the following question: the technological convergence and catch up that indeed have had an impact on economic growth, can suggest that the *VIDE*'s coefficient of heredity (Research and Development for this model) will determine the trajectory of countries' patents? In this sense, the research is devoted to study in what way the stability of trajectory of patents, (specifically conditioned by the level of accumulation of *R&D*) helps to understand the process of convergence and/or catching up. This technological variable approach is introduced in one endogenous growth model with transitional dynamics leading towards an analysis to determine and see if technological and economic convergence (sectoral *GDP* per-capita) occurs. We will lean on the specific case of the bio-pharmaceutical sector, known for its high *R&D* spending amounts and its importance regarding patents, between 1980-2008. We have based our study on *USPTO* patents classes 514, 424, 435 and 800 from the bio-pharmaceutical field and we have based our equations on developed and emerging countries (China, India and Mexico).

1. Theoretical Framework

One of the most influent contributions to the modern theory of economic growth is the one of Solow (1956) and Swan (1956); in particular, the acknowledgment of *technologic* progress as an explicative exogenous variable for *PIB* growth, in the long term was decisive for the development of new models of exogenous growth.

We are concerned of how technology and capital are related in such a way that the technological progress of an economy will largely depend on the past history of this relationship. Discoveries and innovations are born from the experience that is accumulated in the production (Arrow, 1962). Romer (1986) then creates a model where growth is sustained in a long-term basis,

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explained by the knowledge spillover. The activities of a firm will affect the knowledge accumulation of another. At an individual level, the decreasing capital returns will not disappear, but the economy as a whole will at least notice constant returns. In these contributions to the theory, the endogenous determinants of a sustained long-term growth rate are fundamental, as patents and $R\&D$.

The measure of technology variable is a rather dark even in modern growth theory. An alternative to it is to use patent statistics as indicators of the level of technological advance (Fagerberg, 1987). The question that comes up when analyzing this is to find out how to take this concept into growth models? The integral equation

$$W(t) = kP + \int_{t_0}^{t} H(t,s)P(s)ds \tag{1.1}$$

It is an example proposed by Volterra (Davis, 1962) to solve the problem of torsion on a rope. A first approximation of the problem considers the relationship between the torsion partner P and the angle of torsion, W, is the lineal equation

$$W(t) = kP \tag{1.2}$$

Where k is constant. Nevertheless, the elastic body has experimented fatigue of previous distortions and hence, it has *inherited,* in some way characteristics from the past. To model the previously presented he considered that the heredity effect could be an integral that sums up the contributions for a period of time $[t_o, t]$, like in (1.1), where $H(t, s)$ the kernel of the integral equations is the heredity coefficient. In a concrete way, two aspects determine the future: the current state time plus the contributions of the past heredity. The level trajectory through a period of time is thus explained by the level of patents through period t, the position of $p(t)$ in the present plus the patents correlation trajectory, and other time factors. The solution behavior of the integro-differential solution by Volterra ($VIDE$) of second class, will determine the trajectory of the patents.

The heredity factor is the sum of past contributions to the creation of knowledge for a determined time period $[t_o, t]$. The growth rate of long-term patents is, then explained by the level of patents in the t period, its position in present time, added to the path of the correlation patents and $R\&D$ in the time frame. A $VIDE$ is suggested in this case to shape patent behavior.

The solution of this equation indicates that in long-term terms, $p(t)$ is always close to zero (Appleby, 2002b; Burton, 2005; Davis 1962). New knowledge lacks long-term dynamics. The speed at which it does depend on if capital is bounded above or not. The $R\&D$ level determines the rhythm in which the production of new knowledge is exhausted. The $p(t)$ trajectory can be exponentially asymptotic if the capital accumulation is limited in case there are no limits to the accumulation of $R\&D$, it can display a sub-exponential behavior (Applebyb, 2002; Murakami, 1991). Following a Romer-type shaping (1986) we found out that there is an endogenous growth with transitional dynamics even under increasing returns to capital. The dynamic of Innovation is crucial in the understanding of technological gaps.

2. Endogenous growth with dynamic technology

In Romer model (1986), the relation externalities associated with knowledge accumulation, generate that optimal competitive balance is not reached. The market balance drives to a level of inversion inferior to the optimal for social inversion. The critiques to this model part from the externalities that favor an explosive growth. Parting from a more general abstraction, we could say, except from the $\alpha+\eta=1$ that there is not such thing as a long-term, sustained and stable growth trajectory, which exists under very restrictive hypothesis. It is said that there is a visible structural instability in these models reflected upon the explosive growth. The idea that backs up the endogenous growth models is simple: the new knowledge is born from previous one; they grow according to the inversion in the $R\&D$. For some it is necessary to back up this idea with works dealing with the history of technology.

2.1 The $A\hat{k}p$ model

This work is considered the application of the heredity principle to the technological catch up problem. The last one refers to the process for which the countries can have a benefit because of the existence of a knowledge production stock available in the rest of the world. When the catch up happens, it is expected that the poorer countries experiment a faster economic growth. This investigation wants to verify if the technological and innovative gaps, specially, the patents level gaps, which are strongly related to the differences between the developed countries and those in development. The means to catch up in a high tech industry for non developed countries could be explained with higher growth rates than those registered in the industrialized countries, this subject to an important development in the industrial and technological capacities. The correlation between $R\&D$ and the patents is a factor that may contribute to explain the dynamics in the technological convergence. Differently explained, it means to understand the performance of the Innovation capacity (the patent capacity in our case). In this situation, the convergence leads to the creation of a virtuous cycle that together with the technological transference and the growth in the productivity, could lead the non-developed countries to a convergence trajectory and a catch up. A crucial point to go through this work is the trajectory that follows the patent production and how it determines if convergence is possible or not. This was mentioned before through the introduction of this work, and it implied assuming that the technology is not a constant as the models (Solow and endogenous) propose.

If the growth of the patent stock (finished projects) in time only depend in the knowledge in the period t, it means that there is not such thing as a heredity effect from the past (the integral in the (1.1) equation. When the expense in $R\&D$ is not considered, it will give as a result that the number of patents per year decreases at a constant rate a[a]. Without new ideas there is no advance, hence $p(t)$ is decreasing

$$\begin{aligned} \dot{p}(t) &= -ap(t) \\ p(t) &= p(0)e^{-at} \end{aligned} \tag{2.1}$$

The last equation in (2.1) shows that in long term the knowledge production will be null. Nevertheless, the heredity factor implies us to consider the past of the trajectory followed by the patents in a period of time $[\tau, t]$, in that sense we have that

$$\dot{p}(t) = -ap(t) + \int_0^t H(t-\tau)\, p(\tau) d\tau \qquad (2.2)$$
$$a > 0$$

In (2.2) the heredity factor $H(t-\tau)$ capital can be $K(t)$, the inversion in $R\&D$, a constant or a discount rate such as the Romer example (2001). In that same way as in the Volterra model (Davis, 1962), the trajectory of the patents level through a time period is explained by the level of patents in the period t, the position of $p(t)$ in the present plus the trajectory of the patents correlation and other time factors. The behaving of the (2.2) solution will determine the possibility of the catch up. Because the patents are a non-rivalry good, inputs for producing goods and knowledge.

The capital $K(t)$ is destined to the production of goods

$$Y(t) = [K(t)\, p(t)]^{\alpha} [AL(t)^{1-\phi}]^{1-\alpha} \qquad (2.3)$$

And the combination with the available knowledge, $p(t)$, is a source of technical progress

$$A = B[K(t)\, p(t)]^{\phi}, B > 0,\ \phi \geq 0 \qquad (2.4)$$

Where ϕ is the scale parameter of knowledge and $k = K/L$. If $\phi = 0$ and $p(t) = 1$, the equation (2.3), is the function of production Cobb-Douglas; in case of $\phi = 1$ and $p(t) = 1$, we talk about the AK function. The rest of the equations are

$$\dot{K} = sY(t) - \delta K(t) \qquad (2.5)$$

$$\dot{L}(t) = nL(t) \qquad (2.6)$$

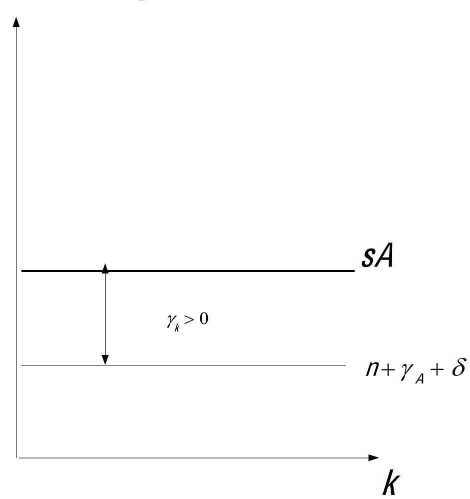
With $n > 0, s > 0, \delta > 0$ y $0 < \alpha < 1$. The equation (2.5) refers to the inversion and the equation (2.6) shaping the work dynamic, and the equations (2.1) to (2.6) shape the economy.

3 Solution to the $\hat{A}kp$ model

3.1 Introduction

The models of endogenous growth are mainly characterized by observing the performance at least not as capital decrement in a long term (Barro and Sala-i-Martin, 1999). In the endogenous growth theory, the long-term capital tendency to present decrement returns must be eliminated. The most elemental of those models is when $Y = AK$ where the medium product and the marginal product for each worker is the same to the A constant. This fact means that the growth rate for any level is k, is $\gamma_k = sA - [n + \delta]$: in a long term, the product per capita can grow even though it does not exist (whatever the reason is) exogenous technical progress, this is $\gamma_A = 0$. The absence of decrement returns of capital is the fundamental property of the exogenous growth models. For some economists it is explained if the concept of capital is expanded in order to include the human capital (Knight, 1944). The distance between the sA parallels and $[n + \delta]$ of graphic 3.1 represents the capital growth rate, γ_k.

Graphic 3.1 *The AK model*



Considering $sA > [n + \delta]$ the capital growth rate is positive and constant, in that case, there is perpetual growth of k even without a technical progress. Accordingly, to this, the balance growth rate γ_k^* is equal to γ_k for any level of k, $\gamma_k = \gamma_k^*$. If we define the consumption per-capita as $c = (1-s)y$, we find out that $\gamma_c^* = \gamma_k^*$. In the same way $\gamma_y = \gamma_k$ for all $t \in [0, \infty)$ because the product per capita is $y = Ak$.

In this way the product, the capital and the consumption per-capita grow at the same rate $\gamma = \gamma^* = sA - [n + \delta]$. The growth rate of the economy depends in the exogenous variables. For example, when the saving rate is elevated or the technology level A, γ^* is raised permanently.

The population growth n or to raise the depreciation δ leads to a contrary effect and a permanent one in γ^*. Contrary to the Solow model (1956), the AK model does not exhibit the property of absolute convergence, neither conditional, translating in $\partial\gamma_y / \partial y = 0$ for all $t \in [0, \infty)$. This result is the principal critique to the model because the empirical evidence in favor of the conditional convergence is important. Our analysis refers to the convergence study when the technology (the number of patents) is modeled by the (2.2) equation.

Before getting to what refers to the convergence again the functions of goods and knowledge production considered in section 2[1]

$$Y(t) = [K(t)\,p(t)]^{\alpha}[\,AL(t)^{1-\phi}]^{1-\alpha}$$

$$A = B[K(t)\,p(t)]^{\phi}, \quad B > 0$$

The production function takes the form of

$$Y(t) = \hat{A}L^{1-\alpha-\phi(1-\alpha)}(pK)^{\alpha}(pK)^{\phi(1-\alpha)} \tag{3.1}$$

and $\hat{A} = B^{1-\alpha}$. When the function is rewritten in per-capita terms when divided between L

$$y(t) = \hat{A}k^{\alpha}\,p^{\alpha+\phi(1-\alpha)}\,k^{\phi(1-\alpha)} \tag{3.2}$$

[1] B is a positive constant that represents displacement in the knowledge function.

We suppose as in the Romer case (1986), that $k(t)$ is the same both in $Y(t)$ as in A. The product per worker is then (intensive form)

$$y(t) = \hat{A}(kp)^{\alpha+\phi(1-\alpha)} \tag{3.3}$$

If the capital accumulation is accordingly to a[2]

$$\dot{K} = sY(t) - \delta K(t) \qquad s > 0, \quad \delta > 0 \tag{3.4}$$

We may consider the fundamental equation in Solow terms with a per-capita population growth and depreciation

$$\dot{k} = sf(k, p) - [n+\delta]k \tag{3.5}$$

Where $f(k, p) = \hat{A}(kp)^{\alpha+\phi(1-\alpha)}$. When substituting the equation (3.3) and dividing by k

$$\begin{aligned} \gamma_k^* &= s\frac{f(k, p)}{k} - [n+\delta] \\ \gamma_k^* &= \frac{\dot{k}}{k} = s\hat{A}k^{-[1-(\alpha+\phi(1-\alpha))]}p^{\alpha+\phi(1-\alpha)} - [n+\delta] \end{aligned} \tag{3.6}$$

On the other hand $f'(k)$ is

$$f'(k, p) = [\alpha+\phi(1-\alpha)]\hat{A}k^{-[1-(\alpha+\phi(1-\alpha))]}p^{\alpha+\phi(1-\alpha)} \tag{3.7}$$

Then $f(k,p)/k = f'(k,p)/[\alpha+\phi(1-\alpha)]$. People speak of endogenous growth if the associated growth rate γ_k^* is positive. In that case, the capital for each worker is not delimited (Barro and Sala-i-Martin, 1999). Simply because γ_k^* is positive when $k \rightarrow \infty$

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2 It is the same as in the Solow model of just one sector of the economy. The product produced product can be used either for consumption or for investment. If $\dot{K} = Y - C$ we define the total consumption of the economy as in $C = (1-s)Y + \delta K$. See Rogers (2003).

$$f(k,p)/k = \hat{A}k^{-[1-(\alpha+\phi(1-\alpha))]}p^{\alpha+\phi(1-\alpha)} > [n+\delta]1/s \qquad (3.7')$$

The equation on the left is the average product, if the limit exists, then

$$\lim_{k\to\infty}\frac{f(k,p)}{k} = \lim_{k\to\infty}\frac{f'(k,p)}{[\alpha+\phi(1-\alpha)]} > \frac{1}{s}[n+\delta] \qquad (3.8)$$

It could be said that the inequality showed before is the necessary and only condition for the model to show endogenous growth in a stable state. If the production function $f(k,p)$ observed decreasing capital output, the marginal product of capital $f'(k,p)$ must be bounded from below so that the condition (3.8) is achieved. There are two variables to consider in this convergence analysis. The first one is the value of ϕ. We assume that it is greater than zero. The other variable is $p(t)$. Let us remember that it is defined by means of the integro-differential equation (2.2)

$$\dot{p}(t) = -ap(t) + \int_0^t H(t-\tau)\,p(\tau)d\tau$$
$$a > 0$$

That reflects the influence of the past knowledge in the creation of new knowledge (patents). The $H(t-\tau)$ kernel is the capital destined to research and development, $K_{R\&D}$. Its accumulation degree is decisive for the convergence dynamics. Let us look at the convergence study.

3.2 Transition to convergence when $p(t) \equiv constant$

The first case is to consider that $\phi < 1$ and p is constant. Given that

$$f'(k) = [\alpha + \phi(1-\alpha)]\hat{A}k^{-[1-(\alpha+\phi(1-\alpha))]}p^{\alpha+\phi(1-\alpha)}$$
$$\frac{f(k)}{k} = \hat{A}k^{-[1-(\alpha+\phi(1-\alpha))]}p^{\alpha+\phi(1-\alpha)}$$

Then

$$\begin{aligned}
&\lim_{k\to\infty} f'(k) = 0 \\
&\lim_{k\to 0} f'(k) = \infty \\
&\lim_{k\to\infty} \frac{f(k)}{k} = 0 \\
&\gamma_k = s\frac{f(k)}{k} - [n+\delta]
\end{aligned} \tag{3.9}$$

This first result is the Solow (1956) one. Of course that, the conditions of Inada[3] are not violated. It exists absolute or relative convergence, given that $\partial\gamma_k/\partial k < 0$ is achieved. Small values of k, correspond to mayor values of γ_k. When $\gamma_y = \gamma_c = [\alpha+\phi(1-\alpha)]\gamma_k$ the analysis of γ_y y γ_c is identical. The next case is when $\phi = 1$ and p is a constant. Where

$$f'(k) = \hat{A}p$$
$$\frac{f(k)}{k} = \hat{A}p$$

Then

$$\begin{aligned}
&\lim_{k\to\infty} f'(k) = \hat{A}p > 0 \\
&\lim_{k\to 0} f'(k) = \hat{A}p > 0 \\
&\lim_{k\to\infty} \frac{f(k)}{k} = \hat{A}p \\
&\gamma_k = s\hat{A}p - [n+\delta]
\end{aligned} \tag{3.10}$$

Like in the AK model the Inada conditions are violated, it exists endogenous growth in a stable state[4], nevertheless neither absolute nor conditional convergence is observed, the same as in the neoclassical model. The convergence property is derived from maintaining the inverse

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[3] Si $\phi < 1$ then $1-(\alpha+\phi(1-\alpha)) > 0$

[4] The necessary and sufficient condition is achieved $\lim_{k\to\infty} \frac{f(k)}{k} > \frac{1}{s}([n+\delta])$

relationship between $f(k)/k$ and k, which does not happen here. If $\hat{A}p > [n+\delta]$ (the growth rate for k is positive for all k) then k grows in a rate of stable state given by $s\hat{A}p - [n+\delta]$. At last, the case where $\phi > 1$ and p is constant

Where

$$f'(k) = [\alpha + \phi(1-\alpha)]\hat{A}k^{-[1-(\alpha+\phi(1-\alpha))]}p^{\alpha+\phi(1-\alpha)}$$
$$\frac{f(k)}{k} = \hat{A}k^{-[1-(\alpha+\phi(1-\alpha))]}p^{\alpha+\phi(1-\alpha)}$$

Then

$$\begin{aligned} &\lim_{k\to\infty} f'(k) = \infty \\ &\lim_{k\to 0} f'(k) = 0 \\ &\lim_{k\to\infty} \frac{f(k)}{k} = \infty \\ &\gamma_k = s[\hat{A}k^{-[1-(\alpha+\phi(1-\alpha))]}p^{\alpha+\phi(1-\alpha)}] - [n+\delta] \end{aligned} \tag{3.11}$$

This result also violates the conditions of Inada. There is no convergence. The growth is explosive. The capital growth rate has a positive relationship with the capital, so that $\partial\gamma_k/\partial k > 0$. A particular matter deserves our attention. Under the initial supposition that p is constant, we retake the equation (2.2) supposing that the kernel is the $K_{R\&D}$ capital. Because of the commutative property of convolution, it's has to

$$\begin{aligned} \dot{p}(t) &= -ap(t) + \int_0^t p(t-\tau)K_{R\&D}(\tau)d\tau \\ 0 &= -a\bar{p} + \int_0^t \bar{p}K_{R\&D}(\tau)d\tau \\ a &= \int_0^t K_{R\&D}(\tau)d\tau \end{aligned} \tag{3.12}$$

When applying the Laplace transform

$$
\begin{aligned}
aL\{1\} &= L\left\{\int_0^t K_{R\&D}(\tau)\right\} \\
a\frac{1}{s} &= \frac{\mathrm{K}(s)}{s} \\
a &= \mathrm{K}(s)
\end{aligned}
\tag{3.13}
$$

When resolving for the capital with the corresponding inverse Laplace transform

$$
\begin{aligned}
L^{-1}[\mathrm{K}(s)] &= aL^{-1}\left[1\right] \\
\frac{K_{R\&D}(t)}{a} &= \delta(t)
\end{aligned}
\tag{3.14}
$$

The inverse Transform of (3.13) is the Dirac Delta defined as

$$
\begin{cases}
\delta(t) = 0, & t \neq 0 \\
\delta(t) = \infty, & t = 0
\end{cases}
\tag{3.15}
$$

The Dirac delta[5] is not a function in the sense that is not defined by calculus books. This one is part of the generalized functions class[6].

To assume that there is no technological change in this case means that the number of patents is constant for a period in the economy. This neoclassical supposition has its own different result in our model. It is observed that before a constant patents trajectory, this meaning the simply reproduction of knowledge, the capital involved in the production of knowledge has an enormous impulse in the initial period $t = 0$, and then it is null for the rest of the period. This result reveals the relationship between technology (number of patents) and $K_{R\&D}$ capital, when the first one is modeled as in (2.2) . This is the big failure in the neoclassical model of perfect competition:

[5] In *The Principles of Quantum Mechanics* published in 1930, the Physics Nobel prize Paul Adrian Maurice Dirac first introduced the Dirac delta.

[6] Given the special characteristics of the Dirac's delta and its importance for the results of this investigation, it is conveniently considered for those that do not know some of its properties to consult the previously cited work. Chapter III, section 15.

thinking that the capital is qualitatively heterogeneous, the relationship with the technology level is of an endogenous character, even though it is supposed to be set and exogenous the orthodox current. Going back to (3.14) and according to the (3.15) definition, the Dirac delta is defined

$$\delta(t-t_0)=\begin{cases} 0, & t\neq t_0 \\ \infty, \lim\limits_{a\to 0}\dfrac{K_{R\&D}(t)}{a}, & t=t_0 \end{cases} \tag{3.16}$$

The particular case where the technological change consists of reproducing the achieved in the last period obliges to invert a great quantity of capital, if increasing the per-capita product is wanted. If it is considered that the capital in the production function is the same class that the one involved in the knowledge production ($R\&D$ in (2.2)), the capital behaves as an impulse function. In the initial period the per-capita product is

$$y_0=\hat{A}(p(t_0)\delta(t-t_0))^{\alpha+\phi(1-\alpha)} \tag{3.17}$$

in addition, the production for any given period is defined as in

$$y(t)=\hat{A}\left[\sum_{i=0}^{n}p(t_i)\delta(t-t_i)\right]^{\alpha+\phi(1-\alpha)} \tag{3.18}$$

Where $\Psi(t)=\sum_{i=0}^{n}p(t_i)\delta(t-t_i)$ it has the formula of an impulse train (Shah function) meaning that

$$y(t)=\hat{A}\left[\Psi(t)\right]^{\alpha+\phi(1-\alpha)} \tag{3.19}$$

This result contradicts the continuity of the neoclassical production function. Supposing that the capital involved in the knowledge production, and the one represented in the production function are differentiated, we would find the same production function as in (3.3) and the capital in (2.2) it is revealed an Arrow-Debreu security (Chance, 2008). In this last point it matches up with the neoclassical model: there are no incentives for the innovation. Nevertheless, the destined capital to produce knowledge exists and it is a continent claim (Arrow-Debreu security). We will return to this in the conclusion of this work.

3.3 Transition to the convergence when $p \equiv p(t)$

Before getting to this section, we will remember some of the results from the integral equation theory related to the solution in the (2.2) equation.

According to Appleby (2002b), the asymptotic behavior of the *VIDE* solutions such as in ($K_{R\&D} \equiv \mathrm{k}$)

$$\dot{p}(t) = -ap(t) + \int_0^t \mathrm{k}(t-\tau)\, p(\tau)\, d\tau \qquad (3.20)$$

A scalar equation of the convolution type, is as followed. Suppose that $a > 0$ and the destined capital to *R&D*, $\mathrm{k}(t)$ [7] is fulfilled as in, $\mathrm{k}(t) : [0, \infty) \to (0, \infty)$, $\int_0^\infty \mathrm{k}(t)\, dt < \infty$ (continuous, integrable and limited, positive and without changing signs) and

$$-a + \int_0^\infty \mathrm{k}(t)\, dt \neq 0$$

In such a case, if

$$a > \int_0^\infty \mathrm{k}(t)\, dt \qquad (3.21)$$

The following equivalent arguments can sustain (Appleby, 2002b)

i) Each solution $p(t)$ in (3.20) converge to zero as $t \to \infty$.
ii) Each solution $p(t)$ in (3.20) is in $L^1[0, \infty)$.
iii) The zero solution is Asymptotically Stable (AS) therefore is Uniformly Asymptotically Stable (UAS)

[7] To simplify the $K_{R\&D} \equiv \mathrm{k}$

In other words (Burton, 2005, theorem 2.6.2) if all solutions $p(t)$ of (3.20) converge to zero as $t \to \infty$ it implies that the condition

$$-a + \int_0^\infty \mathrm{k}(t)dt < 0$$

Is fulfilled, this at the same time leads every $p(t)$ solution is $L^1[0,\infty)$ and the zero solution shows UAS. We arrive to a contradiction when

$$-a + \int_0^\infty \mathrm{k}(t)dt > 0$$

Since it is only fulfilled if for the $p(t)$ solutions from (3.20) , $p(t) \to \infty$ exponentially as $t \to \infty$. In other words, the solutions do not converge. For the case where

$$a = \int_0^\infty \mathrm{k}(t)dt$$

The proposal is as followed. If for every $t \in [0,\infty)$, $t\mathrm{k}(t)$ is associated, so that if $f : t \mapsto t\mathrm{k}(t)$ is integrable in $[0,\infty)$ then the zero solution will US but not UAS. However, if $f : t \mapsto t\mathrm{k}(t)$ is not integrable, the zero solution will have AS but not UAS. We can now make a resume regarding the behaving of (3.20).

Theorem 3.3.1. (First Convergence Theorem)

When $\mathrm{k}(t)$, the kernel in (3.20) , is a positive, continuous and integrable function $\mathrm{k}(t) \in L^1[0, \infty)$, then if

a) $-a + \int_0^\infty \mathrm{k}(t)dt < 0$, Each solution $p(t)$ of (3.20) converges (UAS), $p(t) \to 0$ as $t \to \infty$

b) $-a + \int_0^\infty \mathrm{k}(t)dt > 0$, Each solution $p(t)$ of (3.20) diverges, $p(t) \to \infty$ exponentially as $t \to \infty$.

c) $a = \int_0^\infty \mathrm{k}(t)dt$, Each solution $p(t)$ of (3.20) :

- It is US if $t \mapsto t\mathrm{k}(t)$ is integrable and

- It is AS if $t \mapsto t\mathrm{k}(t)$ is not integrable.

The convergence will be determined for the level of decrease of the number of patents per year, and the quantity of capital that can get involved in the production of new knowledge.

Corollary 1 (Murakami, 1994, Theorem 1) the solution $p(t) \equiv 0$ of (3.20) is UAS and if the condition is fulfilled (necessary and sufficient)

$$\int_0^\infty |\mathrm{k}(t)| e^{\gamma t} dt < \infty$$

For any constant $\gamma > 0$ then is at the same time Exponentially Asymptotically (ExAS). Where $\mathrm{k}(t) \in L^1[0,\infty)$ is exponentially integrable (it is bounded from above) if the condition is fulfilled (Appleby, 2002a). In this way if $p(t) \equiv 0$ is ExAS, then $p(t) \to 0$ is at least exponentially.

Corollary 2. If now $\mathrm{k}(t)$ does not fulfill the necessary and sufficient condition in Corollary 1, meaning that, it is not bounded, What is the convergence rate so that $p(t) \to 0$ as $t \to \infty$ when $\mathrm{k}(t)$ is not anymore exponentially integrable? The convergence rate (Appleby, 2002a, Theorems 6.2 y 6.3) proposed is the following.

Theorem 3.3.2. (Second Convergence Theorem) (Appleby, 2002b and Burton, 2005)

Let $\mathrm{k}(t)$ be a positive *sub-exponential* function[8]. Suppose that $\mathrm{k}(t)$ also meets $-a + \int_0^\infty \mathrm{k}(t) dt < 0$.

Then the solution $p(t)$ of the (3.20) equation satisfies

$$\lim_{t\to\infty} \frac{p(t)}{\mathrm{k}(t)} = \frac{p(0)}{\left[a - \int_0^\infty \mathrm{k}(t) dt \right]^2} > 0, \quad \lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0 \qquad (3.22)$$

[8] See more in Appleby, 2002b. Some of the examples of *sub-exponential* functions are the Pareto, Lognormal and Weibull distributions.

The last example could be the case when $p(t)$ does not converge to zero faster $k(t)$. Moreover, the equation (3.22) implies that $p(t)$ is also a *sub-exponential* function. In which case

$$\lim_{t\to\infty}\frac{p(t)}{p(t)} = \lim_{t\to\infty} -a + \frac{k(t)* p(t)}{p(t)} = 0$$

That is fulfilled when $\lim_{t\to\infty}\dfrac{k(t)* p(t)}{p(t)} = a$ (Appleby, 2002a).

The $k(t)$ kernel is not exponentially integrable. This belongs to the class of *sub-exponential* functions that satisfy $k(t)e^{\gamma t} \to \infty$ condition regarding $t \to \infty$ for every $\gamma > 0$. Therefore $k(t)$ is not limited form above (Corollary 1) and the solution for (3.20) is not an ExAS. It is a *sub-exponential (it decreases slower than an exponential function).*

Before continuing it is necessary to clear up the meaning of (3.22) . It is the capital involved in the knowledge production ($k(t)$ in the (3.20) equation) that determines the velocity in which the number of patents per year decreases. Two possibilities exist: first, if this is bounded from above, $p(t)$ exponentially converges to zero when $t \to \infty$; on the contrary, $p(t)$ converges to zero when $t \to \infty$ but not exponentially, but to the (3.22) rate, which is why it behaves as a *sub-exponential* function.

This being exposed, it exists the possibility to conclude this chapter taking up again the same analysis of convergence made for the case here $p(t)$ is a constant. Once more, we take in consideration the equations (3.1) to (3.8) and (2.2) in order to study the transitional dynamics. Lets start once again defining a product, a marginal product and an average product of the capital for each worker (indirectly it is a function of t)

$$\begin{aligned} y(t) &= f(k,p) = \hat{A}(kp)^{\alpha+\phi(1-\alpha)} \\ \frac{\partial f}{\partial k} &= f'(k,p) = [\alpha+\phi(1-\alpha)]\hat{A}k^{-[1-(\alpha+\phi(1-\alpha))]}p^{\alpha+\phi(1-\alpha)} \\ \frac{f(k,p)}{k} &= \hat{A}k^{-[1-(\alpha+\phi(1-\alpha))]}p^{\alpha+\phi(1-\alpha)} \end{aligned} \qquad (3.23)$$

Theorem 3.3.3. (Third Convergence Theorem)

For $t \to \infty$, $k \to \infty$ y $p \to 0$ and for $t \to 0$, $p \to p(0)$ y $k \to 0$,

$$\text{I) } \phi < 1 \qquad \text{II) } \phi = 1 \qquad \text{III) } \phi > 1$$

$$\lim_{\substack{p\to 0\\ k\to\infty}} f'(k,p) = 0 \qquad \lim_{\substack{p\to 0\\ k\to\infty}} f'(k,p) = 0 \qquad \lim_{\substack{p\to 0\\ k\to\infty}} f'(k,p) = 0$$

$$\lim_{\substack{p\to p(0)\\ k\to 0}} f'(k,p) = \infty \qquad \lim_{\substack{p\to p(0)\\ k\to 0}} f'(k,p) = \hat{A}p(0) \qquad \lim_{\substack{p\to p(0)\\ k\to 0}} f'(k,p) = 0 \tag{3.24}$$

$$\lim_{\substack{p\to 0\\ k\to\infty}} \frac{f(k,p)}{k} = 0 \qquad \lim_{\substack{p\to 0\\ k\to\infty}} \frac{f(k,p)}{k} = 0 \qquad \lim_{\substack{p\to 0\\ k\to\infty}} \frac{f(k,p)}{k} = 0$$

$$\gamma_k = s\frac{f(k,p)}{k} - [n+\delta] \qquad \gamma_k = s\frac{f(k,p)}{k} - [n+\delta] \qquad \gamma_k = s\frac{f(k,p)}{k} - [n+\delta]$$

The first result reflects the model of Solow (1956). The production of knowledge shows the decreasing outputs, $\phi < 1$. The Inada conditions are not violated. It exists absolute or relative convergence, given that $\partial\gamma_k/\partial k < 0$ is fulfilled. Small values of k, correspond to bigger γ_k values. Nevertheless, there is an important difference. The velocity of convergence is conditions- in one part- for the rate in which $p(t) \to 0$ as it was before exposed in the theorems 3.3.1 to 3.3.3. It is the result of the long-term relationship between capital destined to $R\&D$ and the available knowledge. The balance is stable. In other words, the capital is bounded.

The second result is the model $\hat{A}kp$ (version with *AK technology*). The production of knowledge shows constant outputs, $\phi = 1$. Even when the Inada conditions are violated, the convergent trajectory is maintained even though the equation $\partial\gamma_k/\partial k < 0$ is not fulfilled, given that $\partial\gamma_k/\partial k = 0$. This is because $p(t) \to 0$ for any level of capital $k(t)$. The transitional dynamics determines the rates $p(t) \to 0$ and the initial conditions $p(0)$. The balance is stable. The third case is a convergence, too. The production of knowledge shows crescent outputs, $\phi > 1$. Again, the decreased trajectory of $p(t)$, allows the evolution to the convergence. Once again, the conditions of Inada are infringed. In the conditions of crescent outputs of the product, $\partial\gamma_k/\partial k > 0$ and the balance is instable.

3. - Methodology and Data

The empirical analysis of this research is supported by the methodological current of panel data econometrics as below:

1. Multivariate econometric analysis of the patent trajectory set out by the *VIDE*.
2. Multivariate econometric analysis of the growth rate of income per-capita.
3. Convergence sigma analysis according to the coefficient of variation (*CV*).
4. We have based our study on *USPTO* patents kinds 514, 424, 435 and 800 from the bio-pharmaceutical field and we have based our equations on industrial and emerging countries.

4. The Convergence

4.1 Data

The annual guaranteed patent report for each country, in the United States was obtained from the database of the *United States Patent and Trademark Office* (*USPTO*). The research took the types linked o the pharmaceutical industry in the *USPTO*, those corresponding to drugs: 514 (Medicaments and components for the treatment of biological and corporal infections) and 424 (Drugs, bioaffection and the composition of corporal treatments) and those belonging to the biotechnology: 435 (Chemistry: microbiology and molecular) and 800 (Multicellular organisms and the same non-modified parts and related process). The *USPTO* is consulted according to the volume of systematized information that it has, linked to the importance that the *US* has in technology competitiveness.

The data for the added value in the pharmaceutical industry[9] and the fixed brute inversion for each country in their local currency were obtained in the *STAN Database for Structural Analysis (ISIC Rev.3)* 2005 and 2008 of the *OCDE* for the years 1980-2005, except of India[10]. For this

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[9] The added value represents the industry contribution to GDP.
[10] Selected countries: US, Japan, Germany, France, UK, Australia, Belgium, Denmark, NL, Ireland, Mexico, Brazil, Argentina, Korea, China, India and Greece.

country its correspondent *Statistical Yearbook* was consulted. The added value and the fixed brute inversion at common prices were deflected to 1990 prices and converted to dollars so that the 1990 purchasing power parity (*PPP*) related to the *US* could be used. The population for each country was obtained in the *Annual Population Statistics* of the United Nations for several years. The *GDP* per-capita was calculated on the basis of the *GDP* and the population corresponding to each country.

The level of stock knowledge was calculated using the perpetual inventory method (Mohen, 1990):

$$STOCKID = \sum_{\zeta=0}^{3} \delta^{\zeta} (ER\&D_{i,t-\zeta})$$

Where $ER\&D_i$ is the expense in the investigation and the country development i in the t period, in millions of dollars, deflected to 1990 prices. The number of applied regressions is of 3 (ζ = 0, 1,2,3)[11], according to Grilliches (1979). The obsolescence rate is esteemed to be of 15% annually (δ). In the purpose of calculating the capital stock level one must take account the *R&D* level entails that one observation is lost (1980), for which the sample size is reduced.

4.2 Beta convergence

To prove the impact of innovation in the hypothesis of the process of the convergence in the pharmaceutical industry, a model based in the Fagerberg (1987) proposal is developed. If there is a tendency to convergence, the coefficients of the explanatory variables are expected to be negative. The beta convergence is measured by the level of (patents) innovation, as a function of the added value for each previous year worker, the availability of knowledge (level of the previous year patents), and the effort in the *R&D* to innovate, so the position of the pharmaceutical industry of each country is determined according to its technological performance.

[11] This supposition is taken out from the work of Grilliches (1979) in which he argues that the *ER&D* effects persist approximately for 3 or 4 years.

The gaps between the technological level of the pharmaceutical industry amongst the industrialized countries and those in development cause the necessity for other explanatory variables related with the specific conditions of each country.

In this context, the explanatory variables for the technological gaps in the pharmaceutical industry or the convergence tendency of the added value for each worker are considered to be a function of the added value for each worker, the availability of knowledge, and the efforts to innovate. Therefore, three variables are included: I) the added value for each worker in the pharmaceutical industry, with a surplus period ($VALIF_{t-1}$), ii) the number of patents in the last year ($PATIF_{t-1}$), and iii) the investigation surplus stock and the development of the industry in a t period ($STOCKID_t$). According to the model of convergence, we assume that the growth rate of the added value for each worker ($VALIF_t$) is explained by the level from last year ($VALIF_{t-1}$). Besides, if we understand that a country with its own stable economic growth (high PIB per capita) would have more responsibility obtaining positive results from the innovative activities[12], then we assume that the deficit will grow. On the other hand, it is though that the new knowledge, reflected in the level of patents from last year (technological opportunities) is a principal input for the present period innovations.

Finally, the $STOCKID_t$ is the principal input of the innovation[13] and reflects the efforts from the industry to remain in the market. Based on this, the proposed model goes like this:

The model is analyzed through an econometric calculus of panel data that uses the following equation:

$$TVALIF_{it}=\beta_1+\beta_2*LOG(VALIF_{i,t-1})+\beta_3*LOG(PATIF_{i,t-1})+\beta_4*LOG(STOCKID_{it})+e_{it} \qquad [1]$$

Where:

$TVALIF_{it}$ = the growing rate of added value for each worker from the i country in a t period.

[12]This idea is well known as the Schmookler hypothesis. See Schmookler J., Invention and Economic Growth, Harvard University Press, 1966, pp. 28-30.
[13] This happens especially in the pharmaceutical industry.

LOG(VALIF $_{i\text{-}t\text{-}1}$*)*= log of the added value for worker in the pharmaceutical industry from the i country in a t-1 period.

LOG(PATIF $_{i,t\text{-}1}$)= log of the patents level in the pharmaceutical industry from the i country obtained from the *USPTO* (514 or 424 or 435 types and 800 types) in a t -1 period.

LOG(STOCKID$_{it}$)= log of the effort stock in *R&D* from the i country in millions of US dollars 1990 prices in a t period.

e_{it} = error from the i country in a t period.

4.2.1 Empirical evidence

Table 4.2.1 shows of the results from the regression analysis.

Table 4.2.1 Regression analysis of beta convergence (Panel Least Squares)

	1 [*TVALIF*]	2 AR(1) [*TVALIF*]	3 *VIDE* Equation [*DLOG(PATIF)*]
LOG(VALIF$_{i,t-1}$*)*	-0.0223(0.0085)	-0.0172(0.0073)	0.0657(0.0146)
LOG(PATIF$_{i,t-1}$*)*	0.0048(0.0087)	0.0024(0.0073)	-0.0530(0.0142)
LOG(STOCKID$_{it}$*)*	0.0026(0.0072)	0.0038(0.0055)	
Constant	0.0954(0.0367)	0.0749(0.0299)	
R-squared	0.0088(0.1671)	0.0495(0.1658)	0.0303(0.4702)
AR(1)		-0.2021(0.0675)	
	375 observations	360 observations	375 observations

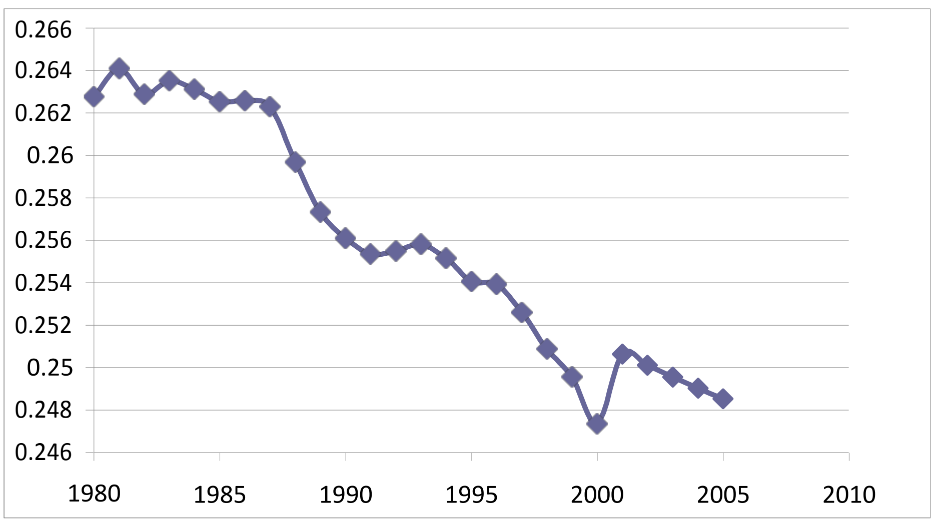
Between brackets it is showed the standard error for the estimate of β and for the R-squared the standard error of the regression. Between the square brackets the dependent variables for each regression are showed. Level of significance of 5%.

For the 1 and 2 regressions the estimate coefficient sign for the added value for each worker (*LOG(VALIF*$_{i,t-1}$*)* is negative. In the typical convergence analysis, this would be a good transitional sign to the stable state of which velocity per period is indicated through the estimated value for β_2 in both regressions. In regression 3, the negative coefficient sign of *LOG(PATIF*$_{i,t-1}$*)* is expected. It is an estimate of the *VIDE* equation that works as a goodness of fit test. It does not pretend more. $VALIF_{i,t-1}$ was used to estimate the convolution integral according to the production function in 3.3 equation.

4.3 Sigma convergence

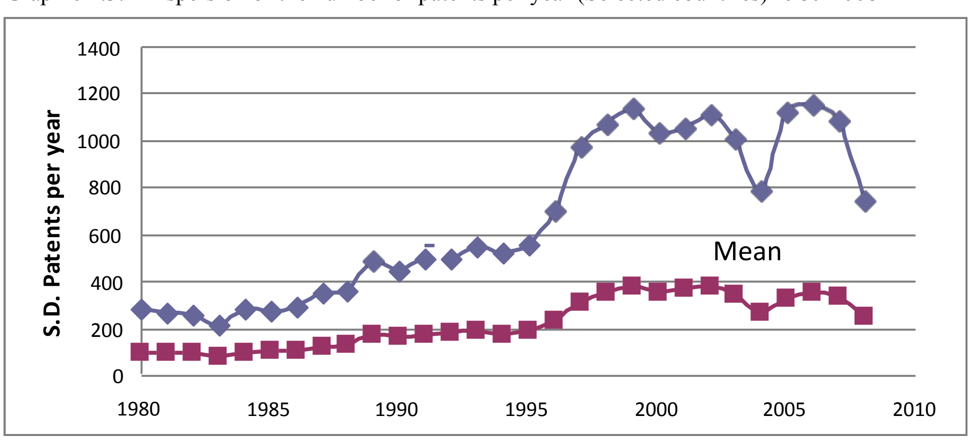
Different from the beta convergence approach, of which analysis focuses on detecting the catching-up evidences and the technological gaps, the goal in studying the sigma convergence is to distinguish if any decrease happens regarding the dispersion between regions or countries (Graphics 4.3.0 and 4.3.1). We want to prove to what extent the number of patents and the *R&D* expense are sub exponential functions, considering that, as it was previously exposed, that the catching-up process depends in the patents relations and the *R&D*, where the second determines the convergence velocity.

Graphic 4.3.0 Coefficient of Variation: Pharmaceutics GDP per capita. (Selected countries) 1980-2008



Source: Own elaboration with *STAN Database (ISIC Rev.3)* 2005 and 2008 *OCDE (1990 prices)*

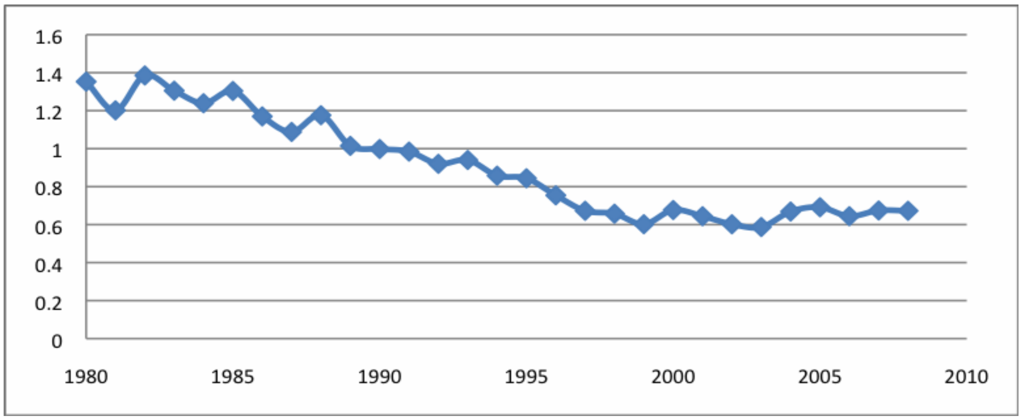
Graphic 4.3.1 Dispersion of the number of patents per year (Selected countries) 1980-2008



Source: Own elaboration with *United States Patent and Trademark Office* (*USPTO*) data

Theoretically speaking, the beta convergence is necessary but not enough for the existence of the sigma. The disadvantage of the first one lies in how we talk of conditional beta, when the economies do not have the same stable state balance[14]. In the case of this study, the evaluation made to the beta convergence can be explained, and even strengthen by the dispersions and the coefficient of variation[15] analysis (Graphic 4.3.2).

Graphic 4.3.2 Coefficient of Variation: number of patents per year. (Selected countries) 1980-2008



Source: Own elaboration with *United States Patent and Trademark Office* (*USPTO*) data

As mentioned in the section before, some of the examples of sub exponential functions are the Pareto, Log-normal and Weibull distributions. The Weibull distribution, for example, is a distribution that depends in two parameters: the form parameter α and the θ[16] scale parameter. A special case happens when the form parameter $\alpha = 1$, that corresponds to the exponential

[14] See Quah 1993.
[15] Defined as $CV = \sigma/\mu$. It is also known as the relative standard deviation. An advantage of the CV use from the standard deviation (SD), is that the second has sense only when the mean is reported. In the case of CV, with a given value of the SD high or low variability is indicated regarding the mean.
[16] It is related with other distributions, for example to the Rayleigh distribution Rayleigh $\alpha = 2$.

distributions function: a characteristic of this function is that its CV is same to 1[17]. The next graphic shows some values for the CV (that is only a function of α) of the Weibull distribution.[18]

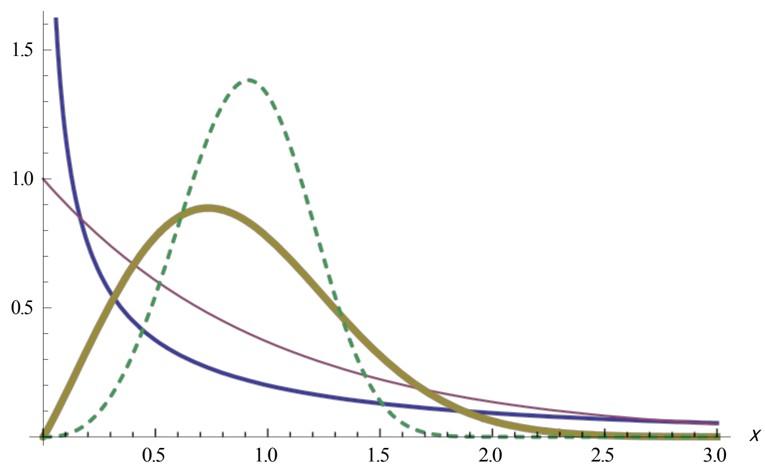
Table 4.3.1 Coefficient of Variation for the Weibull distribution, different values from parameter α

CV	α	CV	α
2.0	0.5427	0.5	2.1014
1.5	0.6848	0.2	5.7974
1.0	1.0000	0.15	7.9069
0.9	1.1128	0.10	12.154

Source: Barringer & Associates, Inc. 1999

As showed in table 4.3.1, when $\alpha = 1$, the $CV = 1$ corresponds to an exponential decay. An inverse relation exists between CV and α. As the variability regarding to the average decreases α increases and the Weibull distribution gets close to the normal distribution when $\alpha = 3.6$ (discontinuous line).

Graphic 4.3.3 Weibull distribution



Source: Own elaboration with Wolfram Mathematica© 8

The *sub-exponential* convergence is then given in a $1 < \alpha < 3.6$ interval, as showed in the 4.3.3 graphic. The discontinuous line is the approximation from Weibull distribution to normal distribution with $\alpha = 3.6$ and $\theta = 1$ parameters. The thicker line is a Weibull with $\alpha = 2.1014$

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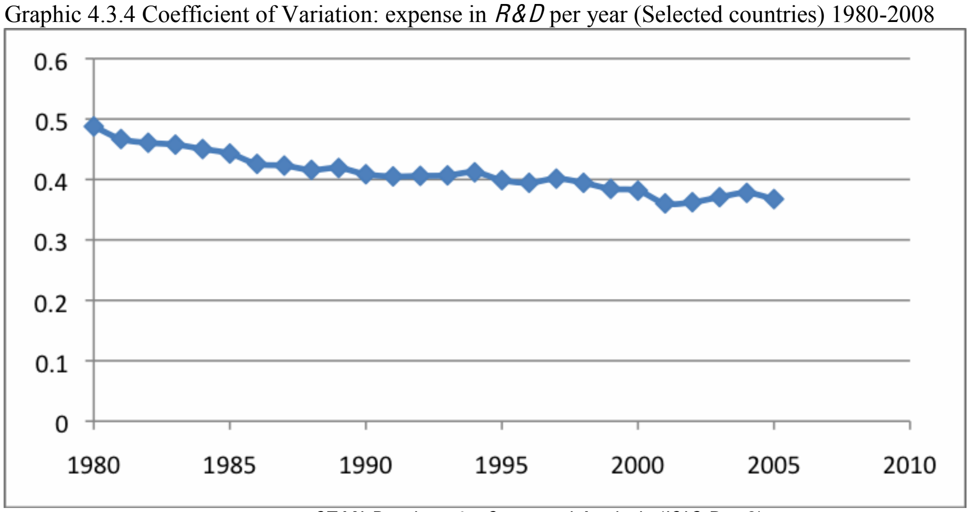
[17] It is the case of the Erlang distribution function where $CV = 1/k^{0.5}$ given that k is a non-negative $CV < 1$. For the gamma distribution where $CV = 1/k^{0.5}$ and $\alpha < 1$, where $CV > 1$.

[18] The distributions where the $CV < 1$ are said to be of low dispersion while others with $CV > 1$ are known to be of high dispersion.

and $\theta = 1$. The last one has the property of a *sub-exponential* function: more slowly than any decaying exponential (The thicker line versus the thinner). The line under the exponential is one with $\alpha = 0.5427$ and $\theta = 1$. Going back to the correspondent graphics of the number of patents dispersion and the variable coefficient, both showed patrons of convergence for the group of countries. In the first one, the volatility is very high in the two thirds of the sample; additionally, the patents production starts to stabilize since the year 2000, even though the dispersion is high. Nevertheless, the standard deviation hides a more relevant aspect from the sigma convergence analysis. As the CV decreases, the convergence slows. Decay less than the exponential function (the thinner line in graphic 4.3.3). Thus $p(t) \to 0$, the zero solution is uniformly asymptotically stable (UAS) but not exponentially asymptotically stable (ExAE).

Graphic 4.3.4 Coefficient of Variation: expense in $R\&D$ per year (Selected countries) 1980-2008



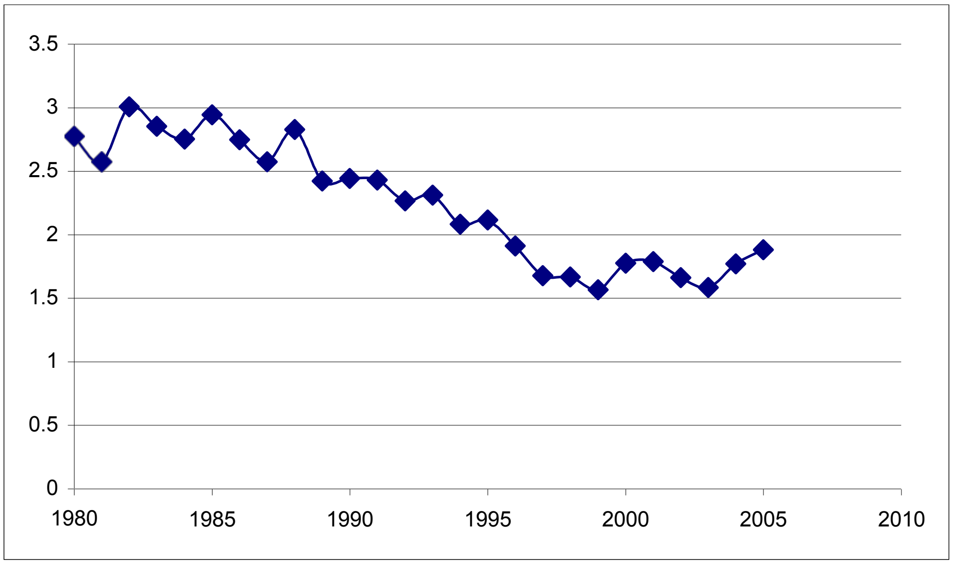
Source: Elaboration with data from *STAN Database for Structural Analysis (ISIC Rev.3)* 2005 and 2008

For the 1980-2005 period, it fluctuates between 0.5 and 0.4. Volatility is marginally decreased. In the same way as in the year's patents, the Weibull distribution associated to the CV drops in a *sub-exponential* dropping zone. According to the second convergence theorem if

$$\lim_{t\to\infty}\frac{p(t)}{k(t)} > 0,$$

the *R&D* and $p(t)$ in 3.20 must be *sub-exponential* functions. The graphic 4.3.5 show the evolution of the $p(t)$ -*R&D* ratio in terms of their coefficients of variation.

p(t)-R&D Coefficient of Variation ratio (Selected countries) 1980-2008



Source: Own elaboration with *United States Patent and Trademark Office* (*USPTO*) data and *Database for Structural Analysis (ISIC Rev.3)* 2005 and 2008.

5. Conclusions

It is considered in this investigation the application of the principle of heredity in Volterra´s (*VIDE)* integro-differential equations to the problem of technological catch-up. This type of equations, suggested by Vito Volterra, acknowledge the "heredity influence" when dealing problems of mechanical and population growth. When modeling with differential equations, variables and its derivates are relational. Deep down, a "non-heredity" principle is taken for granted, when presuming that the future of a system for a determined model depends only from its actual state. This is a restrictive hypothesis, because in occasions the future of a system seems to depend in primary states; this is also the case of heredity. This problem refers to the process in which the countries can get benefited from the existence of a knowledge production stock available in the rest of the world. The principal hypothesis of the correlation between *R&D* and the patents can help us to explain- according to the patents dynamic- that the technological convergence is positive. A central part of this work refers to the trajectory followed by the patents production, and how it contributes to a possible convergence scenario when technological progress is introduced to an endogenous growth model as in Romer (1986) and doing the convergence analysis of the per-capita income variables, the capital for each worker, and *R&D* in these investigations.

The technology modeling is not yet conclusive in actual theory. The alternative was using the patents statistics as an indicator of the degree of technological development. To use this concept in the growth models, we worked with the already exposed Volterra´s idea. The future behavior of the knowledge production, patents per year $p(t)$, not only depends on the position of $p(t)$ in t_0, meaning that, the present state also depends on what it has *inherited* from the past.

The heredity factor is the sum of the past contributions and the creation of knowledge for a time of $[t_0, t]$. The growth rate from the patents level in a long term are explained because of the patents level in t, its position in the present, plus the trajectory of the patents correlation in *R&D* through time. By means of *VIDE,* this behavior is modeled. The solution for this equations indicates that in a long-term, $p(t)$ approaches always approaches to zero. There is a transition to a long-term stable state balance, due to the asymptotical behavior of the solution. The new knowledge looses impulse in the long-term trajectory. The velocity in which it is done depends on, if the capital is limited from above or not. A fundamental discovery is that the level of *R&D* determines the rhythm in which the production of new knowledge is exhausted. In other words, the accumulative process of capital distortions the natural trajectory of the innovation exhaustion. The trajectory of $p(t)$ can be exponentially asymptotically if the accumulation of capital is limited. If there are no accumulation limits of *R&D* -for instance, an absolute position of leadership- can have a *sub-exponential* behavior (distortional). It is warned that we are heading slowly to a path of stable state that is not enough. The decreasing outputs of the innovation also loose their rhythm. There is endogenous growth with transitional dynamics still under crescent outputs, even though the place is unstable, both when productive capital is exhausted, and when it is accumulated, in the same way as in Romer (1986).

A particular case deserves our attention: assuming that the number of patents per year is constant. This implies that the destined capital to $R\&D$, exists in a phase of searching for new inversions, and the degree of innovation looses its force through time, as the inversion emigrates. The analysis of convergence throws the same results as in the Solow-Swan model. Nevertheless, the kernel ($R\&D$) satisfies the $VIDE$ proposal, that is the Dirac delta. This functional form is the one from the Arrow-Debreu securities, and it is fundamental ground for the modern financial theory. This result is robust in the supposition that the involved capital is equally qualitative, and then one must reconsidered the functional form of the production function to a discreet one, the Dirac delta, that leaves behind the results from the micro economy theory of a well-behaved function, and the theoretical consequences deriving from this. A global vision of the results reveals that the innovation process is structured in three stages. The first one is the $R\&D$ capital accumulation, which permits to consolidate an absolute leadership position, process known in theory as entrance barrier. Secondly, the inversion stops flowing at the same intensity in consequence of a fall in the production of new knowledge or because of the emergency for new and more profitable paradigms, encouraging a more just market structure; this is recognized as a convergence stage. The growth rates depend from the initial position of the involved capital in the production, and the number of patents per year in this initial period. Finally, the third stage closes de cycle, and it is characterized for being here the technological trajectories tend to become stagnant, and the $R\&D$ starts a speculative ($R\&D$ is an Arrow-Debreu type security).

On the technological side, the distribution functions such as the Weibull, contribute to distinguish the complex convergence behavior of $R\&D$ capital and the number of patents per year. A process of convergence exists such as the Barro-Sal-i-Martin, beta and sigma type analysis show, unless the present analysis show an asymptotic decreasing behavior in the catch-up controlled by the $R\&D$ capital. In the case of Barro-Sal-i-Martin the behavior is always exponentially asymptotically stable (ExAS). The expressed proposal in this work shows that a complex relation exists between capital-technology, and the beta convergence must be considered as a particular case where $R\&D$ is well limited.

We underline one last aspect: the possibilities of analyzing the same problem form a stochastic sphere. In this case, the solution for a stochastic $VIDE$ solution could enrich the low-uncertainty analysis, and also with the extreme value theory, because of the importance of *sub-exponential* distributions in that field of investigation.

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